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LETTER TO THE EDITOR

On the absence of directed fractal percolation

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Abstract. In the context of the fractal percolation process, the possibility of directed percolation is raised: Is there any non-trivial value of the parameters such that there is 'oriented percolation' or, perhaps 'stiff percolation' in this model? Somewhat surprisingly, the answer is no.

The fractal percolation process (Mandelbrot percolation) was invented in [1, 2] and has been described in a number of places (see, e.g., [3] or [4]). A brief definition is as follows. Let $N \geqslant 2$ denote an integer, let $Q \in [0, 1]$ and let A_0 denote the unit square $[0, 1]^2$. In the first stage, A_0 is divided into N^2 equal-sized squares, each of which is independently *retained* with probability Q or *discarded* with probability $(1 - Q)$. The set $A_1 \subset A_0$ is defined as the (union of the closure of the) squares that were retained. Next, $A_2 \subset A_1$ is obtained by performing the analogous procedure—scaled down in size by a factor of $\frac{1}{N}$—on all the squares of A_1 and, in a similar fashion, one generates $A_{k+1} \subset A_k$. The principal object of interest is the limiting set

$$A_\infty = \bigcap_k A_k. \tag{1}$$

The sets A_k can also be regarded as subsets of $\{1, 2, \ldots, N^k\}^2 \equiv \Lambda_k$. The notions of connectedness in Λ_k are defined in the fashion that is usual for $\mathbb{Z}^2$. For example, a connected path in Λ_k is a sequence of points $((x_1, y_1), \ldots, (x_m, y_m))$ in Λ_k such that $|x_{j+1} - x_j| + |y_{j+1} - y_j| = 1$, $j = 1, \ldots, N^k - 1$. Percolation, for this model, is defined as follows: Let Θ_k denote the event

$$\Theta_k = \{A_k | A_k \text{ contains a connected path between the left and right sides of } [0, 1]^2\} \tag{2}$$

and let $\vartheta_k(Q) = P_Q(\Theta_k)$ denote the probability of Θ_k. Since $\Theta_k \supset \Theta_{k+1}$, the (ϑ_k) are monotone. Let $\vartheta_\infty(Q) = \lim_{k\to\infty} \theta_k(Q)$. Percolation is said to occur whenever $\vartheta_\infty(Q)$ is positive. The percolation threshold is defined via

$$Q_c = \sup\{Q | \vartheta_\infty(Q) = 0\}. \tag{3}$$

In [5] it was established that $1 > Q_c > 0$ for all $N > 1$.

In the same way, one can define various sorts of directed percolation in the fractal percolation model. Let $\mathcal{P}$ denote a connected path in Λ_k. The path $\mathcal{P}$ is said to be *stiff* if, for all j, $x_{j+1} \geqslant x_j$ and $\mathcal{P}$ is said to be *NE-oriented* if, for all j, $x_{j+1} \geqslant x_j$ and $y_{j+1} \geqslant y_j$. The events Ψ_k and Φ_k may be defined for stiff and oriented crossings respectively as in equation (2) and similarly one has $\psi_k(Q)$ and $\varphi_k(Q)$ and their limits. Finally, Q_s may be defined as in equation (3) as the threshold for stiff percolation while Q_o is defined as the threshold for oriented percolation. The principal result of this letter is

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Theorem A. For all N, the stiff and oriented thresholds for the fractal percolation process are given by $Q_s = Q_o = 1$.

Remark. (i) Theorem A has additional consequences in the context of Mandlebrot percolation. In particular, it was shown in [6] that in the region $Q_c \leqslant Q < Q_s$ the dimension of any crossing path in A_∞ is greater than one. Evidently this is all of $[Q_c, 1)$. This, in turn, has ramifications for models of spin-systems in aerogels (see, e.g., [7]) that are currently under investigation.

(ii) The result is somewhat surprising, at least to the author, who in particular is responsible for a widespread rumour to the contrary of theorem A.

Although it is obviously the case that $Q_s = 1 \Rightarrow Q_o = 1$, it is easier to first show that $Q_o = 1$ and, as an immediate corollary to the methods, prove that $Q_s = 1$. The theorems and proofs will be divided accordingly into parts A_o and A_s.

The proof of theorems A. For $L > 0$ consider the fractal percolation process on $[0,1] \times [0,L]$. If L is an integer, this is just L independent copies of the process on $[0,1]^2$. If L is not an integer, let us adhere to the rule that any square with a non-zero fraction both inside and outside of $[0,1] \times [0,L]$ is automatically retained. Let $A_k^{[L]}$ be the notation for a generic kth level configuration on $[0,1] \times [0,L]$, let $\Lambda_k^{[L]}$ denote the corresponding kth level lattices and let $\Psi_k^{[L]}$ denote the event that at the kth level, there is an NE-oriented path up through $[0,1] \times [0,L]$:

$$\Psi_k^{[L]} = \{A_k^{[L]} | A_k^{[L]} \text{ contains a NE-oriented path connecting } [0,1] \times \{0\} \text{ with } [0,1] \times \{L\}. \tag{4}$$

Assuming full retention for the first $k-1$ iterations of the process, the kth level has the appearance of an NE-oriented percolation problem on an $N^k \times LN^k$ lattice. As such, for any fixed $Q < 1$, it is straightforward to show (cf proposition 1 later) that if L gets too large, the probability of such crossings tend to zero with an exponential rate proportional to N^k.

Remark. An optimal proof of this statement would require a detailed investigation of dual contours. For oriented and stiff site percolation problems on $\mathbb{Z}^2$ it is convenient to go to a *bond* representation of the configuration: an oriented bond is occupied iff both the sites at its endpoints are occupied. Contours can then be defined on the dual lattice, according to the standard description, e.g. in [8]. This is tedious for site problems and, in the multiscale versions, somewhat unwieldy. For the purposes of this letter, such complications will be avoided because the only contours that are used are simple enough to be verified by inspection.

Definition. A simple contour on $\Lambda_k^{[L]}$ is a sequence of squares $S_1, \ldots, S_R$ of various scales, $|S_1|, \ldots, |S_R|$ with lowest-leftmost corners represented by the lattice points $(x_1, y_1), \ldots, (x_R, y_R)$ such that

(0) The S_l are commensurate with the multiscale lattice structure, that is in units where the lattice spacing in $\Lambda_k^{[L]}$ is unity, $|S_l|$ is of the form N^t, $t \leqslant k$ and (x_l, y_l) is of the form (aN^t, bN^t) with a and b positive integers not in excess of N^{k-t} and LN^{k-t}. (Explicitly, these squares are located where the correspondingly sized vacant squares of the process are allowed to be.)

(1) $x_i + |S_i| = x_{i+1}$, $i = 1, \ldots, R-1$. (Explicitly, the right edge of the ith square is aligned with the left edge of the $i+1$st.)

(2) $y_i \leqslant y_{i+1} + |S_{i+1}|$, $i = 1, \ldots, R-1$. (Explicitly, the top of the $i+1$st square is not below the bottom of the ith.)

(3) $x_1 = 1$ and $x_R + |S_R| - 1 = N^k$. (Every vertical line runs into the contour.)

A glance at figure 1 will elucidate the important features listed above. A generic simple contour for the kth level will be denoted by $C_{k,\alpha}$ with α running over some appropriate (L and k dependent) index set. It is clear that if each square of a contour $C_{k,\alpha}$ is vacant, the event $\Psi_k^{[L]}$ is precluded. By abuse of notation, the former event will also be denoted by $C_{k,\alpha}$ and the event that at least one such $C_{k,\alpha}$ occurs will be denoted by $C_k^{[L]}$:

$$C_k^{[L]} = \bigcup_\alpha C_{k,\alpha}. \tag{5}$$

Finally, noting that a kth level contour implies a $k+1$st level contour, consider the event

$$C_\infty^{[L]} = \bigcup_k C_k^{[L]} \tag{6}$$

that there is a contour at some level. In the following, it is (easily) demonstrated that for L sufficiently large, $C_\infty^{[L]}$ occurs with probability one. In a certain sense, this is close to theorem A_0 which is the statement that $C_\infty^{[1]}$ occurs with probability one.

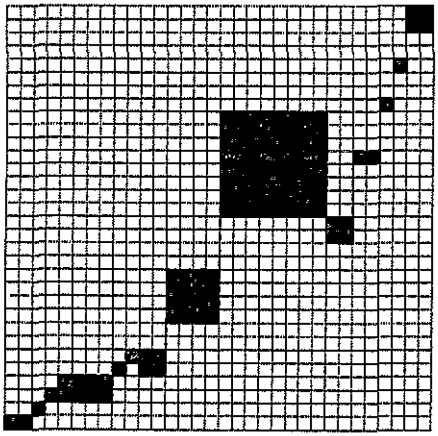

Figure 1. A simple contour on $[0, 1] \times [0, 1]$. Here $N = 3$ and $k = 5$.

Proposition 1. Let $N > 1$, $Q < 1$ and consider the process on $[0, 1] \times [0, L]$ as described. Then for L sufficiently large, $C_\infty^{[L]}$ occurs with probability one.

Proof. Let us investigate the probability of observing a $C_{k,\alpha}$ on $\Lambda_k^{[L]}$ that uses only the kth level squares. To simplify matters, let us further restrict attention to squares $S_1, \ldots, S_{N^k}$ for which $y_{i+1} \geqslant y_i$. The entire contour can be described by the height differences $\Delta_i = y_i - y_{i-1}$, $i = 2, \ldots, N^k$; $\Delta_1 \equiv y_1 - 1$. If the problem is considered on the infinite strip of width N^k (i.e. $\Lambda_k^{[\infty]}$) then the Δ_i are independent and identically distributed geometric random variables:

$$P_Q(\Delta_i = n) = Q^n(1 - Q). \tag{7}$$

A contour occurs on $\Lambda_k^{[\infty]}$ with probability one, a contour occurs on $\Lambda_k^{[L]}$ provided that the 'unconstrained' Δ_i satisfy $\sum_i \Delta_i < N^k L$. The mean height of the unconstrained variables is $N^k Q/(1-Q)$. Thus, if $L > Q/(1-Q)$, as $k \to \infty$, a restricted contour of the type described occurs with probability tending to one exponentially with a rate proportional to N^k. Evidently, in the multi-scale problem, these restricted contours are, with probability one, present on all but a finite number of levels. □

Proof of theorem A_0. The stated result is established by showing that for any $L > 0$, the quantity $C_\infty^{[L]}$ has probability one. This will be accomplished by showing that if for some L, $C_\infty^{[L]}$ has probability one, then so does $C_\infty^{[L']}$ where L' is any number that satisfies

$$L' > QL. \tag{8}$$

It turns out that if L is an integer, the proof captures the essence of the argument and is devoid of a number of spurious details. Let us therefore run through this case first and afterwards fill in the extra steps required for non-integer L.

Suppose then that L is an integer and consider the $N^k \times LN^k$ lattice, $\Lambda_k^{[L]}$. Here, and in what is to follow, let us set the unit scale to the lattice spacing on $\Lambda_k^{[L]}$. Now consider only vacancies on the unit scale and smaller. If r additional iterations of the process occur, each $1 \times L$ rectangle on $\Lambda_k^{[L]}$ appears to be a miniature copy of $\Lambda_r^{[L]}$. (Notice that in fact an expanded configuration space is actually under consideration. In particular, the presence or absence of vacancies sitting below larger scale vacancies is noted in this larger space.) Hence, as $r \to \infty$, the probability of the analogue of $C_r^{[L]}$ in any one of these rectangles tends to one. Hence, for any δ with $1 \gg \delta > 0$, let us rest assured that some $r = r(k)$ has been chosen large enough so that in all the $1 \times L$ rectangles, in $\Lambda_{k+r}^{[L]}$ (with half integer coordinates) the appropriate analogue of the event $C_r^{[L]}$ has occurred with probability in excess of $1 - \delta$.

For ease of exposition, let us now revert to a continuum description—the discrepancy is half a unit—so the lattice $\Lambda_k^{[L]}$ will now be regarded as unit squares that tile $[0, N^k] \times [0, LN^k]$. If only the rectangles $[0,1] \times [0,L]$, $[1,2] \times [L,2L], \ldots$ are considered—which stack up corner to corner—then with probability larger than $1 - \delta$, the event $C_{k+r}^{[L]}$ has been achieved. This, however, represents nothing ventured, nothing gained. The situation is vastly improved by taking into account the effects of vacancies on the unit scale. Working these vacancies in tandem with the collection of all the $1 \times L$ rectangles with 'built in' contours on the smaller scales, it is straightforward to produce a simple contour with a height above the jth column, H_j that is typically a great deal less than jL.

The first step is to examine the square $[0,1] \times [0,1]$ and see if this houses a unit scale vacancy. If not, let us observe the contour event that is the analogue of $C_r^{[L]}$ in $[0,1] \times [0,L]$ and proceed to the square $[1,2] \times [L-1,L]$. On the other hand, if the square $[0,1] \times [0,1]$ is vacant, the contour is off to a good start and the next step is to examine the square $[1,2] \times [0,1]$. Whichever outcome occurs, the procedure outlined, as translated appropriately, starts anew from the stated square. Thus, on the jth step, if the height of the current contour is H_j, look at the unit square $[j, j+1] \times [H_j - 1, H_j]$ and see if this is vacant. If not, tack on a $1 \times L$ rectangle in the jth column in the form of the scaled down version of $C_r^{[L]}$ in $[j, j+1] \times [H_j, H_j + L]$ (recall that this is guaranteed with conditional probability one) and it is concluded that $H_{j+1} = H_j + L$. However, if $[j, j+1] \times [H_j - 1, H_j]$ is vacant, this square is adjoined to the contour and $H_{j+1} = H_j$. The procedure is illustrated in figure 2. It is noted that this 'contour height process' boils down to adding up independent increments:

$$H_{j+1} = H_j \qquad \text{with probability } (1-Q)$$
$$H_{j+1} = H_j + L \qquad \text{with probability } Q \tag{9}$$

with $(H_{j+1} - H_j)$ mutually independent for all j. The contour is completed when $j = N^k$ and the ultimate rescaled height, H_{N^k}/N^k averages to QL. Since this average is the result of independent and identical steps (with only two possible states) the rescaled height is smaller than any number in excess of QL with probability tending to one as k tends to infinity. If for any reason this programme has failed, e.g. for k large, say this happens with probability less than 2δ, the whole game may be started anew $(r+k)$th level of the fractal percolation process. Evidently, if $L' > QL$, the event $C_\infty^{[L']}$ has probability one.

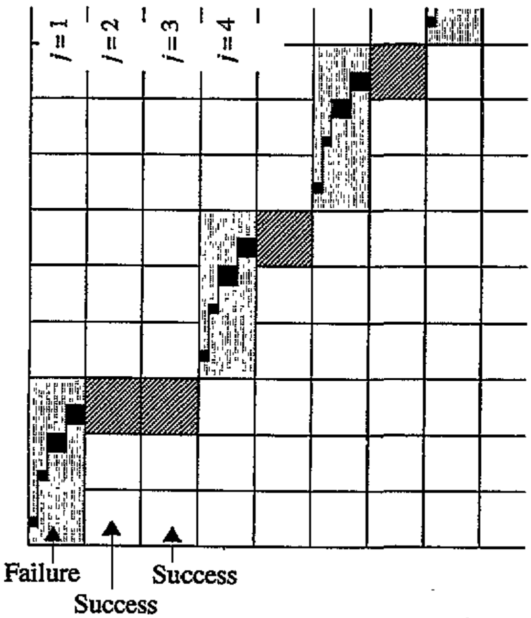

Figure 2. The 'height process'. Here, $L = 3$. Notice that a more efficient process could have been derived in which after each success, the subsequent column height lowered by one. For non-integer values of L, partial decrease in the column height is permitted after a success (although this advantage is thrown away in the bound).

The case of non-integer L is slightly complicated due to the non-commensurability of the $1 \times L$ rectangles and the 'lattice' $\Lambda_k^{[L]}$ but is otherwise almost identical. Let s and r denote integers with $r > s$. Consider the approximate $1 \times L$ rectangles in $\Lambda_k^{[L]}$ that are of the form

$$aN^{-s} \leqslant y \leqslant aN^{-s} + [LN^s]N^{-s} \qquad b \leqslant x \leqslant b+1$$

where a and b are integers and $[LN^s]$ is the largest integer smaller than LN^s. Thus, for large s, these rectangles are ever so slightly larger than $1 \times L$'s but they have the advantage that they lock into the existing lattice structure. Let us denote their height by $\tilde{L}$. Within one of these rectangles, let us consider r iterations of the process below the unit scale (as set by $\Lambda_k^{[L]}$). Since, by hypothesis, $C_\infty^{[L]}$ has probability one, for r large, the probability of observing a scaled down version of $C_r^{[L]}$ in every one of these rectangles exceeds, let us say, $1 - \delta$.

At this point, a height process may be constructed that is nearly identical to the one described in the integer case. Here we will also allow for the possibility of placing unit-scale vacancies slightly below the present height of the contour. Thus, if H_j is the height of the contour in the jth column, the game is to look at the unit square in the $j+1$st column with a height above the x axis of $([H_j]-1)^+$—the largest non-negative integer smaller than H_j. If the said square is vacant, then $H_{j+1} = ([H_j]-1)^+$. On the other hand, if it is occupied, the height increases by the amount $\tilde{L}$ by the tacking on of a $1\times\tilde{L}$ in the $j+1$st column. This process is (stochastically) bounded by the process

$$\begin{aligned} K_{j+1} &= K_j \qquad \text{with probability } (1-Q) \\ K_{j+1} &= K_j+\tilde{L} \qquad \text{with probability } Q \end{aligned} \tag{10}$$

and the argument proceeds in pretty much the same fashion as in the case of integer L. □

Proof of theorem A_s. Let k be an integer and let a be an integer between 1 and N^k-2. Consider the region

$$T_k(a) = \{x, y \in [0,1]^2 | 0 \leqslant x \leqslant 1, (a-1)N^{-k} \leqslant y \leqslant (a+2)N^{-k}\}. \tag{11}$$

Viewed as a subset of Λ_k, the region $T_k(a)$ is just an $N^k \times 3$ rectangle. Let us divide $T_k(a)$ into a top, middle and bottom third and denote the top and bottom regions by $T_k^+(a)$ and $T_k^-(a)$ respectively. Let $\mathcal{T}_k^+(a)$ denote the event that there is a vacant simple contour in $T_k^+(a)$ and, similarly, let $\mathcal{T}_k^-(a)$ denote the event that a reflection of a simple vacant contour (i.e. one that blocks SE-oriented crossings) occurs in $T_k^-(a)$. It is not hard to see that the event $\mathcal{T}_k^+(a)$ does not allow any occupied stiff path starting on the segment $\{0\}\times[a, a+1]$ to rise above the line $y = a+2$ and similarly, the event $\mathcal{T}_k^-(a)$ prevents any such path from falling below the line $y = a-1$ (cf figure 3).

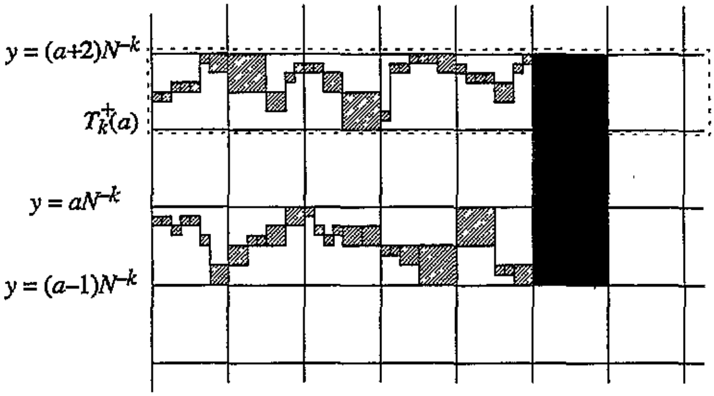

Figure 3. The absence of stiff percolation.

As a consequence of the result derived in the proof of theorem A_o, the events $\mathcal{T}_k^+(a)$ and $\mathcal{T}_k^-(a)$ both have probability one. Suppose that, in addition, there are three vertically stacked squares of side N^{-k} in the region $T_k(a)$ that are all vacant. As illustrated in figure 3, this 'blocking wall' joins with the contours in $T_k^+(a)$ and $T_k^-(a)$ and is seen to prevent any stiff path emanating from $\{0\}\times[a, a+1]$ to reach the line $x = 1$. The probability of a blocking wall is $1-(1-Q)^{3N^k}$ thus if k is large, there is a blocking wall in each $T_k(a)$ with probability close to one. From this it follows easily that for any $Q < 1$, $\varphi_\infty(Q) = 0$. □

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